(because CW-complexes $\subset$ Top_{cof} = retracts of cell complexes)

We now discuss the functoriality of model categories.

Def 53: Let M, N be model categories.

- A Quillen adjunction is an adjunction

$$F : M \underset{\perp}{\rightleftarrows} N : G$$

such that $\begin{cases} F(Cof) \subset Cof \\ G(Fib) \subset Fib \end{cases}$

Lemma 54: $F \dashv G$ is a Quillen adjunction

$$\Updownarrow$$

$F(Cof) \subset Cof$ and $F(Cof \cap W) \subset Cof \cap W$

$$\Updownarrow$$

$G(Fib) \subset Fib$ and $G(Fib \cap W) \subset Fib \cap W$.

proof: Exercise.

Lemma 55: Let $F : M \rightleftarrows N : G$ be a Quillen adjunction. Then there is an induced derived adjunction

$$\mathbb{L}F : \underset{\substack{\text{is} \\ M[W^{-1}]}}{\mathrm{Ho}(M)} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \underset{\substack{\text{is} \\ N[W^{-1}]}}{\mathrm{Ho}(N)} : \mathbb{R}G$$

such that

- $\mathbb{L}F(X) \simeq F(X_{cof})$
- $\mathbb{R}G(Y) \simeq G(Y_{fib})$.

$\square$

This, applied to model categories like $\mathrm{Cpl}_R^{(\pm)}$, is the source of derived functors in homological algebra.

Def 56: Let $F : M \rightleftarrows N : G$ be a Quillen adjunct°. It is a Quillen equivalence if the derived adjunction $\mathbb{L}F \dashv \mathbb{R}G$ is an equivalence of categories.

(this is equivalent to: for $X \in M_{cof}$ and $Y \in N_{fib}$, then

$$F(X) \longrightarrow Y \text{ in } W_M \iff X \to G(Y) \text{ in } W_N)$$

Many functors we have seen in this course turn out to be part of Quillen adjunctions or equivalences.

Thm 57: (Quillen) The adjunction

$$|-| : \text{sSet} \rightleftarrows \text{Top} : \text{Sing}$$

(with Kan-Quillen) (with Quillen)

is a Quillen equivalence.

Sketch: • Quillen adjunction: We check that $|\text{Cof}| \subset \text{Cof}$ and $\text{Sing}(\text{Fib}) \subset \text{Fib}$.

- The first part, we know because of the (relative) skeletal filtration for monomorphisms of simplicial sets. $\Rightarrow |\mathrm{Cof}| \subseteq \text{relative CW} \subseteq \mathrm{Cof}$.

- The second part is easy because of the following:

$$
\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d] & \mathrm{Sing}(X) \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed, "?"] & \mathrm{Sing}(Y)
\end{tikzcd}
\quad \Longleftrightarrow \quad
\begin{tikzcd}
D^{n-1} \simeq |\Lambda^n_k| \arrow[r] \arrow[d] & X \arrow[d] \\
D^{n-1} \times I \simeq |\Delta^n| \arrow[r] \arrow[ur, dashed, "?"] & Y
\end{tikzcd}
$$

which shows that $\mathrm{Sing}(\text{Serre fibrat}^{\circ}) \subset \text{Kan fibrat}^{\circ}$.

- <u>Quillen equivalence:</u> This is a theorem of Milnor which we already mentioned in previous discussions of simplicial homotopy theory.

See [Kerodon, § 3.5]. $\square$

This in some sense the final version of the representation of homotopy types by simplicial sets.

Thm 58: (Lurie) The adjunction

$$\mathrm{Path}[-] : sSet \rightleftarrows Cat_{\Delta} : N_{\Delta}$$

(with Joyal) (with Bergner)

is a Quillen equivalence. $\square$

- This is the promised "equivalence of homotopy theories" between two models of $(\infty, 1)$-categories.

- We now briefly mention simplicial model categories.

Def 59: A simplicial model category M is both a simplicial category and a model category such that:

- M is tensored and cotensored over sSet

- (SM7) for every cofibration $i : A \hookrightarrow B$ and fibration $p : X \twoheadrightarrow Y$,

the simplicial pullback-hom

$$\mathrm{Hom}_M(B, X) \longrightarrow \mathrm{Hom}_M(A, X) \underset{\mathrm{Hom}_M(A,Y)}{\times} \mathrm{Hom}_M(B, Y)$$

is a Kan fibration, which is

trivial if either i or p is. $\square$

Ex 60: • $\mathrm{sSet}_{\text{Kan-Quillen}}$, with its self-enrichment coming from the cartesian closed structure, is a simplicial model category (this is proven with the same "lifting calculus" methods that we saw in Chapter III)

• CGHaus has a natural structure of

simplicial category with

CGHaus is cartesian closed.

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Sing}\big(\underbrace{\underline{\mathrm{Hom}}(X,Y)}_{\in\ \mathrm{CGHaus.}}\big)$$

and the Quillen-type model structure on CGHaus is a simplical model category.

- ⚠ $sSet_{Joyal}$ is <u>not</u> a simplicial model category in this sense, because it has fibrations which are not Kan fibrations.

Def 61: Let M be a simplicial model category. Then, because of (SM7), the full simplicial subcategory on M_{cf} is locally Kan. The ∞-category associated to M is $N_\Delta(M_{cf})$. $\square$

Ex: . If $M = sSet_{\text{Kan-quillen}}$, then

$M_{cf} = \widetilde{Kan}$, and $N_\Delta(\widetilde{Kan}) =: Spc$.

. $M = CGHaus_{\text{quillen}} \rightsquigarrow$

$$Spc' := N_\Delta(CGHaus_c)$$

. From $|\cdot| : sSet \rightleftarrows CGHaus_{\text{quillen}} : Sing$ quillen eq., we can get

$$Spc \underset{\sim}{\rightleftarrows} Spc'.$$

Thm 62: (Simpson, Dugger, Lurie)

Let C be an ∞-category.

The following are equivalent:

- C is a presentable ∞-category

($\mathcal{C}$ admits all colimits in ∞-categorical sense, and is generated under filtered colimits by "small" objects)

- There exists a combinatorial simplicial model category M such that $\mathcal{C} \simeq N_\Delta(M_{cf})$.

$\square$

This theorem explains a posteriori the success of the theory of model categories: they "model" many interesting ∞-categories.

V Joins, slices, (co)limits

1) The 1-categorical story

Let's start with the piece of category theory which we want to generalize to ∞-categories.

Def 1: Let C be a category.

(1) Let X be an object of C. The slice category over category $C_{/X}$ is defined as the pullback

$$\begin{tikzcd} C_{/X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1], C) = \mathrm{Ar}(C) \arrow[d, "ev_1"] \\ \{X\} \arrow[r] & C \end{tikzcd}$$

- The coslice category under category $C_{X/}$ is defined dually (using ev_0).

Concretely, an object of $C_{/X}$ is a morphism $Y \to X$, and a morphism of $C_{/X}$ is a commutative triangle

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr] & \arrow[d, phantom, "="] & Z \arrow[dl] \\ & X & \end{tikzcd}.$$

$$C_{/X} \longrightarrow C \longleftarrow C_{X/}$$

$$\begin{array}{c} Y \\ \downarrow \\ X \end{array} \longmapsto Y \qquad Z \longleftarrow\!\shortmid \begin{array}{c} X \\ \downarrow \\ Z \end{array}$$

(2) Let J be another category and $F : J \longrightarrow C$ a functor.

Let $C \xrightarrow{\underline{(\,)}} \mathrm{Fun}(J, C)$ be the "constant functor" functor.

The slice category $C_{/F}$ is defined as

$$C \underset{\mathrm{Fun}(J,C)}{\times} \left(\mathrm{Fun}(J,C)_{/F} \right).$$

i.e the pull back

$$\begin{tikzcd} C/_F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\mathcal{J},C)/_F \arrow[d] \\ C \arrow[r, "\underline{(\ )}"] & \mathrm{Fun}(\mathcal{J},C) \end{tikzcd}$$

i.e. its objects are the cones over F: an object $X \in C$ together with a natural transformation $\underline{X} \overset{\lambda}{\Longrightarrow} F$, so a collection of morphisms $(X \xrightarrow{\lambda_i} F(i))_{i \in \mathcal{J}}$ such that, for every morphism $f: i \longrightarrow j$, we have

$$\begin{tikzcd} & F(i) \arrow[dd, "Ff"] \\ X \arrow[ur, "\lambda_i"] \arrow[dr, "\lambda_j"'] \arrow[r, phantom, "\equiv"] & \\ & F(j) \end{tikzcd} .$$

The coslice category $C_{F/}$ is defined dually as the category of cocones under F.

Def 2: Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$.

Then X is an initial (resp. terminal) object of $\mathcal{C}$ if the canonical functor
final

$$\mathcal{C}_{X/} \longrightarrow \mathcal{C} \quad \left(\text{resp. } \mathcal{C}_{/X} \longrightarrow \mathcal{C}\right)$$

is an equivalence of categories.

Exercise 3: Check this is equivalent to your favourite definition of initial/terminal.

Def 4: Let $F : \mathcal{J} \longrightarrow \mathcal{C}$ be a functor.

A limit (resp. a colimit) of F is an terminal (resp. initial) object in $\mathcal{C}_{/F}$

(resp. $C_{F/}$).

Exercise 5: • Again, check that this fits with your favourite definition of (co)limit.

• Check using these definitions that an initial object is the same thing as

- a colimit of the empty functor $\emptyset \to C$
- a limit of the identity functor $C \to C$.

The charm of these definitions is that everything is built from slice categories. So if we can construct "slice ∞-categories" we can hope to define limits and colimits.

- The problem is that in the ∞-categorical context, we don't want the triangles

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr, "f"'] & \arrow[d, phantom, "="] & Z \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

to commute on the nose but up to coherent homotopy.

It turns out to be easier to define the left adjoint of the (co)slice construction.

Def 6: Let C, D be categories. The **join** $C * D$ of C and D is the category defined as follows:

- $\mathrm{Ob}(C * D) = \mathrm{Ob}(C) \amalg \mathrm{Ob}(D)$

- $$C * D(X, Y) = \begin{cases} C(X,Y), & X, Y \in C \\ D(X,Y), & X, Y \in D \\ *, & X \in C, Y \in D \\ \emptyset, & X \in D, Y \in C \end{cases}$$

and composition is defined in the obvious way.

- The join defines a functor:

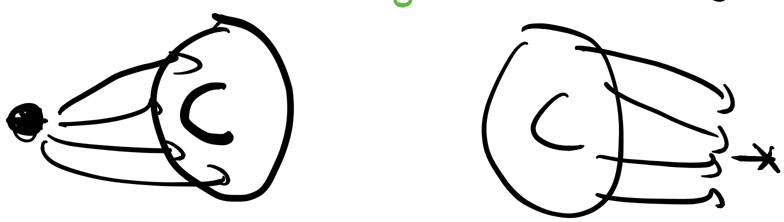
$$- * - : \mathrm{Cat} \times \mathrm{Cat} \longrightarrow \mathrm{Cat}$$

- By construction, there are fully faithful functors

$$C \xrightarrow{\iota_C} C * D \xleftarrow{\iota_D} D.$$

- We define the left cone $C^{\triangleleft} := [0] * C$

  right cone $C^{\triangleright} := C * [0]$.



Rmk 7: $*$ defines a (non-symmetric) monoidal structure, with monoidal unit $\emptyset$.

- $C \longrightarrow C^{\triangleleft}$ is "the universal way to add an initial object to C", as we will see later.

Lemma 8: Let C, D be categories.

(1) The functor ι_C factors uniquely as

$$C \xrightarrow{\bar{\iota}_C} (C * D)/_{\iota_D} \longrightarrow C * D$$

(2) The functor ι_D factors uniquely as

$$D \xrightarrow{\bar{\iota}_D} (C * D)/_{\iota_C} \longrightarrow C * D$$

proof: By definition,

$$(C * D)/_{\iota_D} = (C * D) \times_{\mathrm{Fun}(D, C*D)} \mathrm{Fun}([1] \times D, C * D) \times_{\mathrm{Fun}(D, C*D)} \{\iota_D\}$$

so a factorisation as in (1) is the same thing as a functor

$$C \longrightarrow \mathrm{Fun}([1] \times D, C * D)$$

satisfying some properties, and by adjunction the same as a functor

$$C \times D \times [1] \longrightarrow C * D,$$

that is a natural transformation

from

$$\begin{tikzcd} & C \arrow[dr, "\iota_C"] & \\ C \times D \arrow[ur, "\pi_C"] \arrow[rr, "\iota_C \circ \pi_C"'] & & C * D \end{tikzcd}$$

$$\iota_C \circ \pi_C : C \times D \longrightarrow C * D$$

to $\iota_D \circ \pi_D : C \times D \longrightarrow C * D$.

There is a <u>unique</u> such transformation υ, given by $(x, y) \in C \times D \rightsquigarrow$ the unique element of $C * D(x, y)$. $\square$

Prop 9: Let $\begin{cases} C \text{ be a category} \\ G : D \to E \text{ be a functor.} \end{cases}$

There is a bijection

$$\left\{ \begin{tikzcd} & D \arrow[dl, "\iota_D"'] \arrow[dr, "G"] & \\ C * D \arrow[rr, dashed] & & E \end{tikzcd} \right\} \cong \mathrm{Cat}(C, E_{/G}).$$

$$U \longmapsto \bar{F}(U)$$

given as follows:

$$\begin{tikzcd} & D \arrow[dl, "\iota_D"'] \arrow[dr, "U \circ \iota_D"] & \\ C * D \arrow[rr] & & E \end{tikzcd}$$

For every functor $U : C * D \to E$ such that that $D \xrightarrow{\iota_D} C * D \to E$ is equal to G, let $\bar{F}(U)$ denote the composite

$$C \xrightarrow{\bar{\iota}_C} C * D_{/\iota_D} \xrightarrow{U} E_{/U \circ \iota_D} = E_{/G}$$

Dually, we have:

$$\underbrace{\left\{ \begin{tikzcd} & C \arrow[dl, "\iota_C"'] \arrow[dr, "F"] & \\ C * D \arrow[rr, dashed] & & E \end{tikzcd} \right\}} \cong \mathrm{Cat}(D, E_{F/}).$$

Fun

proof: Omitted. See [Kerodon, Prop 4.3.2.10 and Prop 4.3.2.13] $\square$

Rmk: The proof shows that we have in fact a pushout square in Cat:

$$\begin{tikzcd}
(C \times \{0\} \times D) \amalg (C \times \{1\} \times D) \arrow[r] \arrow[d] & C \times [1] \times D \arrow[d, "v"] \\
(C \times \{0\}) \amalg (\{1\} \times D) \arrow[r] & C \ast D \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Cor 10: A colimit of $F: I \to C$ is the same thing as a functor $\hat{F}: I^{\triangleright} \to C$ extending F and initial for this property.

proof: By Prop. , we have that

$$\mathrm{Ob}(\mathcal{C}_{F/}) \simeq \left\{ \begin{tikzcd} & I \arrow[dl] \arrow[dr, "F"] & \\ I^{\triangleright} = I * [0] \arrow[rr] & & \mathcal{C} \end{tikzcd} \right\}$$

This actually lifts to an equivalence of categories

$$\mathcal{C}_{F/} \simeq \left\{ \begin{tikzcd} & I \arrow[dl] \arrow[dr, "F"] & \\ I^{\triangleright} = I * [0] \arrow[rr, "\hat{F}"] & & \mathcal{C} \end{tikzcd} \right\}$$

A colimit of F is an initial object of $\mathcal{C}_{F/}$; by looking at initial objects on the right, the result follows. $\square$

$$\left(\hat{F}(\text{cone pt}) = \operatorname{colim} F, \quad \begin{matrix} i \to \text{cone} \\ F(i) \to \operatorname{colim} F \end{matrix} \right)$$

Cor 11: • For any category D, the functor

$$\mathrm{Cat} \longrightarrow \mathrm{Cat}_{D/}, \quad C \longmapsto C * D$$

is the left adjoint to

$$\mathrm{Cat}_{D/} \longrightarrow \mathrm{Cat}, \ (G : D \to E) \mapsto E_{G/}$$

- For any category C, the functor

$$\mathrm{Cat} \longrightarrow \mathrm{Cat}_{C/}, \ D \longmapsto C * D$$

is the left adjoint to

$$\mathrm{Cat}_{C/} \longrightarrow \mathrm{Cat}, \ (F : C \to E) \mapsto E_{F/}.$$

proof: This is just a reformulation of Prop. $\square$

2) Joins of simplicial sets

Def 12: The augmented simplex category Δ_+ is the full subcategory of PoSet spanned by $[-1] = \emptyset$, $\overbrace{[0], [1], \dots}^{\mathrm{Ob}(\Delta)}$.

In other words, we add an initial object to Δ. We can also write

$$\Delta_+ = \Delta^{\triangleleft}$$ using the notation from the previous section.

Def 13: The category of augmented simplicial sets is $sSet_+ := \mathrm{Fun}(\Delta_+^{op}, Set)$.

We write $\Delta^n = y[n]$, $n \geq -1$ for representables.

Lemma: Let $\iota : \Delta \longrightarrow \Delta_+$ be the inclusion functor. The precomposition functor $\iota^* : sSet_+ \longrightarrow sSet$

has $\begin{cases} \text{a left adjoint } \iota_! : sSet \to sSet_+ . \\ \text{a right adjoint } \iota_* : sSet \to sSet_+ . \end{cases}$

proof: This is a special case of the functoriality of presheaf categories (see Exercise 2.2). $\square$

Lemma 14: There is an equivalence of categories:

$$sSet_+ \xrightarrow{\sim} \left\{ (X, E, a) \,\middle|\, \begin{array}{l} \cdot X \in sSet \\ \cdot E \in Set \\ a : X \to cE \end{array} \right\}$$

$$\tilde{X} \longmapsto \left(\iota^* \tilde{X}, \tilde{X}_{(-1)}, a_{\tilde{X}}\right) \quad \left[\pi_0 X \to E\right]$$

augmentation

where $a_{\tilde{X}}$ is the collection of maps

$\left(\tilde{X}([-1] \to [i])\right)_{i \geq 0}$.

• Via this equivalence, the functors $\iota_! \dashv \iota^* \dashv \iota_*$ are given by the formules

$$\begin{cases} \iota^*(X, E, a) = X \\ \iota_! X = (X, \pi_0(X), X \to \pi_0(X)) \\ \iota_* X = (X, *, X \to \Delta^0 = *) \end{cases}$$

trivial augmentat°

proof: Exercise.

Lemma 15: The category Δ_+ has a monoidal structure defined by join / ordinal sum:

$m, n \geq -1$,

this is the join of categories in the sense of the previous section

• $[m] * [n] = [m+1+n]$

$\{0<1<\dots<m\} * \{\bar{0}<\bar{1}<\dots<\bar{n}\} = \{0<1<\dots<m<\bar{0}<\bar{1}<\dots<\bar{n}\}$

and monoidal unit $[-1] = \emptyset$.

Def 16: The join of augmented simplicial sets is defined as the free cocompletion

$$* : sSet_+ \times sSet_+ \longrightarrow sSet_+$$

of

$$* : \Delta_+ \times \Delta_+ \longrightarrow \Delta_+ \xrightarrow{y} sSet_+$$

in both variables; i.e the unique colimit preserving in both variables functor such that for all $m, n \geq -1$, we have

$$\Delta^m * \Delta^n = \Delta^{m+1+n}.$$

Rmk: This is a special case of an important general construction in category theory, the Day convolution: any monoidal structure on a category C induces a monoidal structure on $PSh(C)$ in a canonical way.

Def 17: Let J be a totally ordered set.

The set of cuts $Cut(J)$ is the set of decompositions $J = J_1 \amalg J_2$ such that $x < y$ whenever $x \in J_1$ and $y \in J_2$.

initial segment of J.

Lemma 18: Let $\alpha : J \to J'$ be an order-preserving map, and $(J_1', J_2') \in Cut(J')$. There exists a unique $(J_1, J_2) \in Cut(J)$ such that α restricts to maps

$\alpha_1 : J_1 \longrightarrow J_1'$ and $\alpha_2 : J_2 \longrightarrow J_2'$.

Hence $\mathrm{Cut}(-)$ is a contravariant functor on totally ordered sets.

proof: Put $J_i = \alpha^{-1}(J_i')$. This is a cut since α is order-preserving. $\square$

Prop 19: Let $X, Y \in sSet_+$. There is a canonical identification, for $n \geq -1$

$$(X * Y)_n = \coprod_{(J_1, J_2) \in \mathrm{Cut}([n])} X(J_1) \times Y(J_2)$$

$$= \coprod_{\substack{i+1+j=n \\ i,j \geq -1}} X_i \times Y_j$$

Moreover, if $\alpha : [m] \to [n]$ is a map in Δ_+ and we fix a decomposit° $i+1+j = n$, there is an induced decomposit° $i'+1+j' = m$ such that $\alpha^{-1}([i]) \simeq [i']$ and $\alpha^{-1}(i+1+[j]) = i'+1+[j']$. and the induced map

$$(X * Y)_n \xrightarrow{\alpha^*} (X * Y)_m$$

is given by $\coprod_{i+1+j=n} \alpha_i^* \times \alpha_j^*$

with $\begin{cases} \alpha_i : [i'] \to [i] \\ \alpha_j : [j'] \to [j] \end{cases}$ induced by α.

as in Lemma 18.

proof: defines an augmented simplicial set and

- The RHS$^\vee$ commutes with colimits in both X and Y because colimits in presheaf categories are computed objectwise, so it suffices to show this for representables

- The point is then precisely that

$$
\begin{aligned}
(\Delta^p * \Delta^q)_n &= (\Delta^{p+1+q})_n \\
&= \Delta_+([n],[p+1+q]) \\
&= \coprod_{i+1+j=n} \Delta_+([i],[p]) \times \Delta_+([j],[q])
\end{aligned}
$$

(where $[i]$ is the preimage of $[p]$ under a map $[n] \to [p+1+q]$.)

$$= \coprod_{i+1+j} \Delta^p_i \times \Delta^q_j$$

This finishes the proof. $\square$

Alternative formulat°:

By definition as free cocompletion, we get

$$(X * Y)_n = \operatorname*{colim}_{E_n} (X_p \times Y_q)$$

where E_n is the category of elements

of the functor $([p],[q]) \in \Delta_+^2 \longrightarrow \Delta_+([n], [p]*[q])$

But every arrow $f : [n] \longrightarrow [p] * [q]$ is

uniquely of the form $u * v$ with $u : [i] \to [p]$

and $v : [j] \to [q]$ by Lemma 18.

So the inclusion $\coprod \{ [i] * [j] \to [p] * [q] \} \hookrightarrow E_n$

is cofinal, and the colimit reduces to

the claimed coproduct.

Def 20: The *join* of simplicial sets is defined as the functor

$$- * - : sSet \times sSet \longrightarrow sSet$$
$$(X, Y) \longmapsto \iota^*((\iota_* X) * (\iota_* Y))$$

Rmk:

- Note that ι_* preserves representables, $\iota_* \Delta^n = \Delta^n$ ($n \geq 0$):

For all $k \geq -1$,

$$sSet_+(\Delta^k, \iota_* \Delta^n) = sSet(\iota^* \Delta^k, \Delta^n)$$

$$= \begin{cases} sSet(\emptyset, \Delta^n), & k = -1 \\ sSet(\Delta^k, \Delta^n), & k \geq 0 \end{cases}$$

$$= \Delta_+([k], [n])$$

$$= sSet_+(\Delta^k, \Delta^n).$$

So $\Delta^m * \Delta^n = \Delta^{m+1+n}$ both in $sSet$ and $sSet_+$

Also, the (non-representable) empty simp. set $\emptyset$ satisfies $L_* \emptyset = \Delta^{-1}$.

- By combining the formulas above, one gets that we still have a formula:

$$(X * Y)_n = \coprod_{\substack{i+1+j=n \\ i,j \geq -1}} X_i \times Y_j$$

with by convention $X_{-1} = Y_{-1} = pt.$,

or alternatively as

$$(X * Y)_n = X_n \amalg \coprod_{\substack{i+1+j=n \\ i,j \geq 0}} (X_i \times Y_j) \amalg Y_n$$

- We have $(X * Y)^{op} \simeq Y^{op} * X^{op}$.
- $(X * Y)_0 = X_0 \amalg Y_0$

$(X * Y)_1 = X_1 \amalg (X_0 \times Y_0) \amalg Y_1$

...